J. Math. Anal. Appl. 383 (2011) 55–70

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Linear differential equations and multiple zeta values. II. A generalization of the WKB method ☆

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ARTICLE INFO

Article history:
Received 17 June 2010
Available online 5 May 2011
Submitted by S. Fulling

Keywords:
WKB expansion
Stokes operator

ABSTRACT

For linear differential equations $x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0$ (and corresponding linear differential systems) with large complex parameter λ and meromorphic coefficients $a_j = a_j(t;\lambda)$ we prove existence of analogues of Stokes matrices for the asymptotic WKB solutions. These matrices may depend on the parameter, but under some natural assumptions such dependence does not take place. We also discuss a generalization of the Hukuhara–Levelt–Turritin theorem about formal reduction of a linear differential system near an irregular singular point $t = 0$ to a normal form with ramified change of time to the case of systems with large parameter. These results are applied to some hypergeometric equations related with generating functions for multiple zeta values.

1. Introduction

This paper is directly related with the papers [22,20,21]. There some relations between ODEs with a parameter and so-called multiple zeta values was observed and studied. Here we develop some tools which are used in [20] and [21].

Recall that the quantity

$$\zeta(a_1,\ldots,a_k) = \sum_{0<n_1<\cdots<n_k} \frac{1}{n_1^{a_1}\ldots n_k^{a_k}}, \tag{1.1}$$

where $a_1,\ldots,a_k$ are positive integers and $a_k \geqslant 2$, is called a *multiple zeta value* (see [19]). Following [22] we construct the following generating functions for multiple zeta values:

$$\begin{aligned} f_{a_1,\ldots,a_k}(\lambda) = 1 &- \zeta(a_1,\ldots,a_k)\lambda^a + \zeta(a_1,\ldots,a_k,a_1,\ldots,a_k)\lambda^{2a} \\ &- \zeta(a_1,\ldots,a_k,a_1,\ldots,a_k,a_1,\ldots a_k)\lambda^{3k} + \cdots, \end{aligned} \tag{1.2}$$

where $a = a_1 + \cdots + a_k$. Consider the following differential operators:

$$\begin{aligned} &\partial_t = \partial/\partial t, \qquad R = (1-t)\partial_t, \qquad Q = t\partial_t, \\ &P = RQ^{a_1-1}RQ^{a_2-1}\ldots RQ^{a_k-1}. \end{aligned} \tag{1.3}$$

It was proved in [22] that

$$f_{a_1,\ldots,a_k}(\lambda) = \varphi_1(1;\lambda),$$

☆ Supported by Polish MNiSzW Nos. N N201 271035 and N N201 397937.

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doi:10.1016/j.jmaa.2011.05.004

where $x = \varphi_1(t; \lambda)$ is the solution to the differential equation

$$Px + \lambda^a x = 0 \tag{1.4}$$

such that $\varphi_1(t; \lambda)$ is analytic in t near $t = 0$ and $\varphi_1(0, \lambda) = 1$.

In particular, the generating function $f_2(\lambda)$ associated with $\zeta(2)$ takes the form

$$f_2(\lambda) = 1 - \zeta(2)\lambda^2 + \zeta(2,2)\lambda^4 - \cdots = \prod\left(1 - \frac{\lambda^2}{n^2}\right) = \frac{\sin(\pi\lambda)}{\pi\lambda}. \tag{1.5}$$

Here Eq. (1.4) is the hypergeometric equation

$$(1-t)\partial_t t \partial_t x + \lambda^2 x = 0 \tag{1.6}$$

with the solution

$$x = \varphi_1(t; \lambda) = {}_2F_1(\lambda, -\lambda; 1; t).$$

Similarly, the generating function $f_3(\lambda) = 1 - \zeta(3)\lambda^3 + \zeta(3,3)\zeta^6 - \cdots$ leads to the third order hypergeometric equation

$$(1-t)\partial_t t \partial_t t \partial_t x + \lambda^3 x = 0. \tag{1.7}$$

The generating function f_2 was computed in (1.5) using the known product expansion for the function sin. However the aim of [20] was to calculate $f_2(\lambda)$ directly from the hypergeometric equation using the so-called WKB method and Stokes operators. In fact, two new proofs of the formula (1.5) are given in [20]. An analogous analysis in [21] leads to existence of a sixth order linear ODE satisfied by $f_3(\lambda)$ near $\lambda = \infty$.

After the above introduction let us pass to the subject of the present paper. We study differential equations

$$x^{(n)} + a_1(t; \lambda)x^{(n-1)} + \cdots + a_n(t; \lambda)x = 0 \tag{1.8}$$

and differential systems

$$\dot{x} = A(t; \lambda) \cdot x, \quad x \in \mathbb{C}^n, \tag{1.9}$$

with a parameter $\lambda \in \mathbb{C}$ which is close to infinity.

When one assumes that

$$a_j = \lambda^j a_{j,0}(t) + O\left(\lambda^{j-1}\right) \tag{1.10}$$

in Eq. (1.8), then it is natural to look for solutions in the following form

$$x = \omega(t; \lambda) = e^{\lambda S(t)} \cdot \left\{ \frac{\psi_0(t)}{\lambda^\gamma} + \frac{\psi_1(t)}{\lambda^{\gamma+1}} + \cdots \right\}, \tag{1.11}$$

where the 'action' function $S(t)$ has derivative $\dot{S}(t) = s(t)$ which satisfies the 'Hamilton–Jacobi equation'

$$s^n + a_{1,0}(t)s^{n-1} + \cdots + a_{n,0}(t) = 0. \tag{1.12}$$

The 'amplitudes' $\psi_k(t)$ satisfy a system of linear first order 'transport equations' of the form

$$\dot{\psi}_k + W(t)\psi_k = Z_k(t), \quad k = 0, 1, \ldots, \tag{1.13}$$

where $W(t)$ is uniquely defined by means of the coefficients $a_{j,0}(t)$ and the non-homogeneous terms Z_k are determined via $\psi_0, \ldots, \psi_{k-1}$. We do not specify the exponent γ at this moment. We note also some trouble associated with choice of constants of integration of the system (1.13), but we skip it in this paper.

In the case of system (1.9) with

$$A = \lambda \cdot \operatorname{diag}\big(b_1(t), \ldots, b_n(t)\big) + O(1) \quad (\lambda \to \infty) \tag{1.14}$$

one expects solutions of the form (1.4), but with $\psi_k(t)$ being vector valued functions.

Analogous solutions we introduced by G. Wentzel [18], H. Kramers [10], L. Brillouin [3] and H. Jeffreys [9] for the multidimensional stationary Schrödinger equation $-\Delta\psi + \frac{2}{\hbar^2}(V(x) - E)\psi = 0$. They are called the *WKB solutions* and the analysis of these solutions is called the WKB method.

In [20] and [21] the WKB method was applied to the hypergeometric equations (1.6) and (1.7) for large parameter λ.

G. Birkhoff [1] was the first who proved analyticity of the WKB solutions $\omega(t; \lambda)$ with respect to the parameter λ in sectors $\mathcal{S}$ (about $\lambda = \infty$ in the Riemann sphere $\widehat{\mathbb{C}}$) such that the solutions $s_i(t)$, $i = 1, \ldots n$, to Eq. (1.12) satisfy the condition

$$\operatorname{Re}\bigl(\lambda s_i(t)\bigr) \neq \operatorname{Re}\bigl(\lambda s_j(t)\bigr), \quad i \neq j, \tag{1.15}$$

for real t. In [2] Birkhoff applied this result to the Schrödinger equation.

In other sources which treat the WKB method, like [5] and [6], one proves convergence of the solutions $\omega(t;\lambda)$ under assumption that λ is real (and $\lambda \to +\infty$) and t belongs to a connected domain $\mathcal{D} \subset \mathbb{C}$, such that inequalities (1.15) hold true. Also it is proved that the WKB solutions can be prolonged to the closure $\overline{\mathcal{D}}$ of the domain $\mathcal{D}$, i.e. when some of the inequalities in Eq. (1.15) become equalities for $t \in \partial\mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D}$.

The lines (in the t-plane) where $\operatorname{Re}(s_i(t) - s_j(t)) = 0$ for some $i \neq j$ are called the *Stokes lines* or the *division lines* (depending on a source).

The Stokes' name appears here in connection with so-called Stokes phenomenon [14] which concerns linear differential equations and systems, without essential parameters, but considered in neighborhood of an irregular singular point. Such a system is of the form

$$t^r \dot{x} = B(t)x, \quad t \in (\mathbb{C}, 0), \tag{1.16}$$

where $r \geqslant 2$ and the analytic matrix $B(t)$ is of the form

$$B(t) = \operatorname{diag}(\mu_1, \ldots, \mu_n) + O(t) = B_0 + O(t) \tag{1.17}$$

with the condition

$$\mu_i \neq \mu_j \quad \text{for } i \neq j. \tag{1.18}$$

(An example of such situation is provided by the so-called multidimensional Landau–Zener system $\sqrt{-1}\frac{d\psi}{dx} = (Ax + B)\psi$, $\psi \in \mathbb{C}^n$, where $A = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ and B are constant matrices, see [4]; here the point $x = \infty$ is irregular singular.)

One defines also the *rays of division*

$$R = \bigl\{t\colon \operatorname{Re}\bigl((\mu_i - \mu_j)t^{1-r}\bigr) = 0,\ \arg t = \text{const}\bigr\} \quad (i \neq j). \tag{1.19}$$

The formal solutions to system (1.16) take the form

$$x = \eta_j(t) = e^{P_j(1/t)} \cdot \bigl\{c_{j,0}t^{\delta} + c_{j,1}t^{\delta+1} + \cdots\bigr\}, \quad j = 1, \ldots n, \tag{1.20}$$

where $P_j(1/t) = \frac{\mu_j}{1-r}(1/t)^{r-1} + \cdots$ are polynomials of degree $r-1$ and $c_{j,k}$ are constant vectors. It turns out that expansions (1.20) are asymptotic in some sectors about $t = 0$, but a general solution to system (1.16) cannot be uniquely expressed as a combination of the basic solutions η_j. The coefficients of such a combination undergo jumps when one changes the sector and this is the *Stokes phenomenon*.

The rigorous treatment of the Stokes phenomenon uses the *sectorial normalization theorem* (proved in the book of W. Wasow [17]) which states that for each open sector $\mathcal{S} \subset (\mathbb{C}, 0)$ which contains at most one ray of division there exists a gauge change $x = G_{\mathcal{S}}(t)y$, analytic for $t \in \mathcal{S}$, which reduces system (1.16) to the diagonal system

$$t^r \dot{y} = \operatorname{diag}\bigl(b_1(t), \ldots b_n(t)\bigr) \cdot y; \tag{1.21}$$

in fact, the functions $b_j(t) = \mu_j + O(t)$ can be chosen as polynomials of degree $\leqslant r-1$. If $\mathcal{S}$ and $\mathcal{S}'$ are adjacent sectors then the matrix valued function $G_{\mathcal{S}\mathcal{S}'} = G_{\mathcal{S}}(t)G_{\mathcal{S}'}^{-1}(t)$, $t \in \mathcal{S} \cap \mathcal{S}'$, called the *Stokes operator*, preserves system (1.21). It transforms the basic solutions $y = \tilde{\eta}_j(t) = \exp(\int b_j t^{-r})e_j = e^{P_j(1/t)}e_j$ (where (e_j) is the basis associated with the coordinates (y_j)) to linear combinations of the basic solutions. The matrix $C_{SS'}$ of the latter change is a constant matrix called the *Stoked matrix*. The same matrix acts on the basic solutions (1.21) to the unchanged system (1.12).

One of the aims of this work is to generalize the sectorial normalization theorem to the case of systems (or equations) with large parameter. Note that in the above mentioned Birkhoff theorem as well as in its variant from the Fedoryuk's book [5] we have the assumption (1.15), which corresponds to the property that the sector S from the sectorial normalization theorem does not contain any ray of division. We want a result about normalization of a system with large parameter in domains $\mathcal{U} \subset \mathbb{C}^2$ which include some surface $\mathcal{R}_{ij} = \{\operatorname{Re}(\lambda s_i(t)) = \operatorname{Re}(\lambda s_j(t))\}$ in its interior.

Such result is proved in Theorem 1 from the next section. Because that theorem needs a series of assumptions, we do not state it here. We only note that we restrict ourselves to the case where the time variable t belongs to a neighborhood of the segment $[0, 1)$ and the system has algebraic singularity at $t = 0$. Some parts of the domains $\mathcal{U}$ with the surfaces $\mathcal{R}_{ij}$ are concentrated near the point $t = 0$, where we use approximation of our system by a system with irregular singularity. It turns out that it is possible to arrange the Birkhoff's proof (away from the surfaces $\mathcal{R}_{ij}$) and the Wasow's proof (near $\mathcal{R}_{ij}$'s) in a compatible manner.

The Wasow's book [17] contains a theorem about reduction of equations of the form

$$\ddot{x} - \lambda^2\bigl\{t\bigl(1 + O(t)\bigr) + O\bigl(\lambda^{-1}\bigr)\bigr\}x = 0 \tag{1.22}$$

(as $t \to 0$ and $\lambda \to \infty$) to the Airy equation

$$\ddot{x} - \lambda^2 t x = 0; \tag{1.23}$$

moreover, the change is analytic in t and λ^{-1}. In Theorem 2 (in Section 2) we generalize this theorem, with rather elementary proof.

The assumption (in Theorem 1) that the singularity at the endpoint $t=0$ of the interval $[0,1)$ is algebraic is not accidental. Note that, when we rewrite Eq. (1.6) and diagonalize the matrix in the right hand side, then the leading part of that matrix will be $\lambda/\sqrt{t(1-t)}\cdot\operatorname{diag}(i,-i)$. On the other hand, introduction of the variable $T=\lambda^2 t$ for $t\approx 0$ results in treating the hypergeometric equation (1.6) as a perturbation of a Bessel like equation $\partial_T T\partial_T x + x = 0$ with irregular singularity at $T=\infty$ (see [20]) and Theorem 2 allows to relate the Stokes operators associated with the hypergeometric equation with the Stokes operators related with the Bessel equations. Also for the Bessel like equation formal reduction of the singularity at $T=\infty$ requires the ramified change of the time: $T=(T')^2$. The fact that asymptotic solutions of some second order equations are expressed via Bessel functions was observed by R. Langer [11,12].

A theorem by M. Hukuhara [7], A. Levelt [13] and H. Turritin [16] asserts that the formal reduction of a linear system near an irregular singularity usually needs a ramified change of time (see also [8], [23] and Section 3). Turritin [15] observed that the same phenomenon should take place in the case of equations or systems with large parameter and that the correct asymptotic solutions should have a polynomials in $\lambda^{1/b}$ (with coefficients depending on t) in the exponent. This should take place when the Hamilton–Jacobi equation (1.12) has a solution $s_i(t)$ of multiplicity greater than one; for example, for the Riemann equation

$$t^2(1-t)^2\ddot{x} - 2\lambda t(1-t)\dot{x} + \lambda^2 x = 0. \tag{1.24}$$

Section 4 is devoted to a discussion of this phenomenon.

2. Generalized normalization theorem and generalized Stokes operators

To formulate our result we make some assumptions. We firstly consider systems of the form (1.9), where the entries $a_{ij}(t;\lambda)$ of the matrix A are meromorphic in $\lambda^{-1}\in(\mathbb{C},0)$ (with pole at $\lambda=\infty$) and multivalued holomorphic in t from a connected and simply connected domain $\mathcal{D}\subset\mathbb{C}$ (which contains the real interval $[0,1)$) with regular singularity at $t=0$. We impose also the following additional assumptions.

A1. We have

$$A(t;\lambda) = \lambda t^{-\alpha}\cdot\operatorname{diag}\bigl(b_1(t),\dots,b_n(t)\bigr) + O(1) \quad \text{as } \lambda\to\infty, \tag{2.1}$$

with $1>\alpha=p/q\in\mathbb{Q}$ and with analytic functions

$$b_j(t) = \mu_j + \cdots, \quad \text{as } t\to 0,\ \mu_j\neq 0, \tag{2.2}$$

such that the 'actions'

$$S_j(t) = \int_0^t \tau^{-\alpha} b_j(\tau)\,d\tau, \quad 0<t<1, \tag{2.3}$$

satisfy the following condition: *the* $n(n-1)$ '*action difference lines*'

$$\mathcal{Q}_{ij} : (0,1)\ni t\mapsto \Delta_{ij}(t) = S_i(t) - S_j(t)\in\mathbb{C}, \quad i\neq j, \tag{2.4}$$

are pairwise non-intersecting and with pairwise different directions at $t=0$.

(Note that the $\arg(\mu_i-\mu_j)$ must be pairwise different.)

A2. The entries $a_{ij}(t;\lambda)\in\mathbb{C}\{t^{1/q},\lambda^{-1}\}[t^{-1/q},\lambda]$, i.e. they are expanded in powers of $t^{1/q}$ and λ^{-1}, and any monomial $\text{const}\cdot t^{k/q}\lambda^{-l}$ in this expansion satisfies

$$\begin{aligned} &k\geqslant -K,\\ &l\geqslant -1,\\ &\frac{k}{q-p} + l > \frac{-p}{q-p} - 1 \quad \text{for } (k,l)\neq(-p,-1), \end{aligned} \tag{2.5}$$

where $K\geqslant p$ is some integer. The last inequality in Eqs. (2.5) means that the admissible monomials have higher degree than the term $\lambda t^{-\alpha}$ in Eq. (2.1) with respect to the quasi-homogeneous gradation given by

$$\widetilde{\deg}\bigl(t^{1/q}\bigr) = 1/(q-p), \qquad \widetilde{\deg}\bigl(\lambda^{-1}\bigr) = 1; \tag{2.6}$$

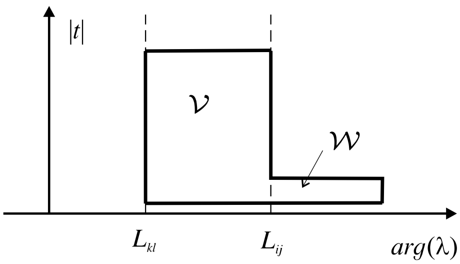

Fig. 1. Scheme of the domain (2.8) for fixed $\arg t$ and $|\lambda|$. The lines L_{ij} and L_{kl} correspond to the surfaces of division $\mathcal{R}_{ij}$ and $\mathcal{R}_{kl}$.

therefore the *fast time*

$$T(t) := \lambda t^{1-p/q} \tag{2.7}$$

has the quasi-homogeneous degree equal zero.

To state main theorem of this section we introduce some domains $\mathcal{U}$ of $\mathbb{C}^2$. Any such $\mathcal{U}$ is associated with a pair of adjacent lines $\mathcal{Q}_{ij}$ and $\mathcal{Q}_{kl}$, i.e. such that between them there are no other action difference lines and $\arg\Delta_{ij}(t) < \arg\Delta_{kl}(t)$. This domain is of the form

$$\mathcal{U} = \mathcal{V} \cup \mathcal{W}. \tag{2.8}$$

Here the set $\mathcal{V}$ is defined by the following inequalities

$$\begin{aligned} &|\arg t| < \delta_1, \qquad |t| < 1/2, \qquad |T| > N, \qquad |\lambda| > \Lambda, \\ &-\frac{\pi}{2} + \delta_2 < \arg\lambda\Delta_{ij}(t) < \frac{\pi}{2} - \delta_2 < \frac{\pi}{2} + \delta_2 < \arg\lambda\Delta_{kl}(t) < \frac{3\pi}{2} - \delta_2. \end{aligned} \tag{2.9}$$

The set $\mathcal{W}$ is associated with the line $\mathcal{Q}_{ij}$ and is of the form

$$\mathcal{W} = \bigl\{|t| < \varepsilon,\ |T| > N,\ |\lambda| > \Lambda,\ \bigl|\arg\bigl(T(\mu_i - \mu_j)\bigr) - \pi/2\bigr| < \gamma\bigr\}. \tag{2.10}$$

Above T is the fast time defined in Eq. (2.7) and the constants Λ, N (large), $\delta_{1,2}$, ε (small) and γ will be fixed later. The domain $\mathcal{U}$ is schematically sketched in Fig. 1, which presents situation in the case with fixed $|\lambda|$ and $\arg t = 0$ and where the lines L_{ij} and L_{kl} roughly correspond to $\arg(\lambda(\mu_i - \mu_j)) = \pi/2$ and to $\arg(\lambda(\mu_k - \mu_l)) = \pi/2$ (or to the 'surfaces of division' $\mathcal{R}_{ij}$ and $\mathcal{R}_{kl}$).

Theorem 1. *There exist the constants Λ, N, δ_1, δ_2, ε, γ such for any domain $\mathcal{U}$ defined above there is a change*

$$x = H_{\mathcal{U}}(t, \lambda, \bar{\lambda}) \cdot y = \bigl(I + O(1/\lambda)\bigr)y, \quad (t, \lambda) \in \mathcal{U}, \tag{2.11}$$

which is analytic in t and real analytic in $(\lambda, \bar{\lambda})$ and which transforms system (1.9) *satisfying assumptions* A1 *and* A2 *to a diagonal system of the form*

$$\begin{aligned} &\dot{y} = \operatorname{diag}(\tilde{b}, \dots, \tilde{b}_n) \cdot y, \\ &\tilde{b}_j(t, \lambda, \bar{\lambda}) = \lambda b_j(t) + O(1) \quad \bigl(|\lambda| \to \infty\bigr). \end{aligned} \tag{2.12}$$

Moreover, in the domain $\mathcal{V}$ defined by the inequalities (2.9) *the matrix-valued function $H_{\mathcal{U}}(t, \lambda, \bar{\lambda})$ is close to a unique function $H^{\mathrm{an}}_{\mathcal{U}}(t, \lambda)$, analytic in (t, λ^{-1}), which follows from the Birkhoff's normalization*:

$$\bigl|H_{\mathcal{U}}(t, \lambda, \bar{\lambda}) - H^{\mathrm{an}}_{\mathcal{U}}(t, \lambda)\bigr| < \mathrm{const} \cdot e^{-\mathrm{const}\cdot|\lambda|}, \quad (t, \lambda) \in \mathcal{V}; \tag{2.13}$$

in particular, the formal expansions in powers of λ^{-1} of the change (2.11) *and the normal form* (2.12) *are uniquely defined and do not depend on the sector $\mathcal{U}$.*

Let us order the domains introduced above counterclockwise, $\mathcal{U}_1, \dots, \mathcal{U}_M$, and take their intersections

$$\mathcal{S}_i = \mathcal{U}_i \cap \mathcal{U}_{i+1}. \tag{2.14}$$

The intersections $\mathcal{S}_i$ are of importance for us, because they contain sector like sets of the form

$$\bigl\{a < t < \varepsilon,\ |\lambda| > \Lambda,\ \alpha < \arg\lambda < \beta\bigr\}, \tag{2.15}$$

i.e. an interval times a sector in the λ-plane; we shall call the sets (2.14) as *sectors*. To simplify notations we write $H_i = H_{\mathcal{U}_i}$.

With any sector $\mathcal{S}_i$ we associate the matrix-valued operator

$$G_i(t;\lambda,\bar{\lambda}) = H_i^{-1}H_{i+1}, \quad (t,\lambda)\in\mathcal{S}_i, \tag{2.16}$$

which we call the *Stokes operator*. We also associate with this sector the basis

$$y = \tilde{\omega}_j(t;\lambda,\bar{\lambda}) = \exp\left(\int_0^t \tilde{b}_j(s;\lambda,\bar{\lambda})\,ds\right)e_j, \quad j=1,\dots n, \tag{2.17}$$

of solutions to system (2.12), where (e_j) is the basis associated with the coordinate system (y_j). The corresponding basis of solutions to the initial system is

$$x = \omega_j(t,\lambda,\bar{\lambda}) = H_i\tilde{\omega}_j, \quad j=1,\dots,n; \tag{2.18}$$

they are the *WKB solutions*, analytic in t and real analytic in $(\lambda,\bar{\lambda})$ in the sector $\mathcal{S}_j$.

Any Stokes operator G_i preserves the diagonal system (2.12). Therefore the vector-valued functions $G_i\tilde{\omega}_j$ are also solutions to system (1.12), i.e.

$$G_i\tilde{\omega}_j = \sum_k c_{jk}^{(i)}(\lambda,\bar{\lambda})\tilde{\omega}_k, \tag{2.19}$$

where the coefficients $c_{jk}^{(i)}(\lambda,\bar{\lambda})$ do not depend on t and form the so-called *Stokes matrices* $C^{(i)}$. Like in the classical Stokes phenomenon, the matrices $C^{(i)}$ are equivalent (via permutation of the basis (2.17)) to a triangular matrix with 1 at the diagonal.

The operators $H_{i+1}H_i^{-1}$ preserve system (1.9) and we have

$$H_{i+1}H_i^{-1}\omega_j = H_{i+1}G_iH_{i+1}^{-1}\omega_j = H_{i+1}G_i\tilde{\omega}_j = \sum c_{jk}^{(i)}H_{i+1}\tilde{\omega}_k = \sum c_{jk}^{(i)}\omega_k; \tag{2.20}$$

i.e. the same Stokes matrices $C^{(i)}$ act on the basic WKB solutions (2.18). Eqs. (2.19) and (2.20) can be interpreted as

$$(\tilde{\omega}_j|\mathcal{U}_{i+1}) = \sum c_{jk}^{(i)}(\tilde{\omega}_k|\mathcal{U}_i), \qquad (\omega_j|\mathcal{U}_{i+1}) = \sum c_{jk}^{(i)}(\omega_k|\mathcal{U}_i), \tag{2.21}$$

where $(\tilde{\omega}_k|\mathcal{U}_i)$ and $(\omega_k|\mathcal{U}_i)$ denote corresponding basic solutions restricted to the domains $\mathcal{U}_i$.

Proof of Theorem 1. As in other normalization proofs in the theory of system of linear ODEs the problem is reduced to transformation of the matrix $A(t;\lambda)$ to the block diagonal form $\left(\begin{smallmatrix} C^{11} & 0 \\ 0 & C^{22}\end{smallmatrix}\right)$, where the submatrices C^{11} and C^{22} correspond to division of the collection $\{b_1,\dots,b_n\}$ of diagonal terms in Eq. (2.1) into two disjoint subsets $\{b_1,\dots,b_m\}$ and $\{b_{m+1},\dots,b_n\}$ (compare [17,23,8]). We seek a change of the form

$$H = \begin{pmatrix} I & H^{12} \\ H^{21} & I \end{pmatrix}.$$

Assuming

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$$

we obtain the following equation for $X = H^{12}$:

$$\dot{X} = A^{11}X - XA^{22} + A^{12} - XA^{21}X; \tag{2.22}$$

an analogous equation holds for H^{21}.

Eq. (2.22) is a particular case of the nonlinear equation

$$\dot{X} = \Theta(t)X + F(t,X), \quad X\in\mathbb{C}^R, \tag{2.23}$$

where $\Theta(t)$ is a diagonal matrix with terms $\theta_r(t) = \lambda t^{-\alpha}(b_{i'}(t) - b_{j'}(t))$ (where $i'\leqslant m<j'$ or $j'\leqslant m\leqslant i'$) at the diagonal. Eq. (2.23) is equivalent to the integral equation

$$X(t) = \int_\Gamma \mathcal{F}(t)\mathcal{F}^{-1}(s)F\big(s,X(s)\big)\,ds, \tag{2.24}$$

where $\mathcal{F}(t) = \operatorname{diag}(\exp\int\theta_1,\dots,\exp\int\theta_R) = \operatorname{diag}\{\exp\lambda(S_{i'}-S_{j'})(t)\} = \operatorname{diag}\{\exp\lambda\Delta_{i'j'}S(t)\}$ is the fundamental matrix of the linear part of Eq. (2.23) and $\Gamma = (\Gamma_1,\dots,\Gamma_R)$ is a collection of paths in the s-plane each of which ends up at $s=t$.

Recall that in each component of the vector equation (2.24) the integration takes place along corresponding path Γ_r. The point is that the paths Γ_r should be chosen in such a way that the factors

$$\exp\int_s^t \theta_r(\tau)\,d\tau = \exp\bigl\{\lambda\bigl(\Delta_{i'j'}(t)-\Delta_{i'j'}(s)\bigr)\bigr\},$$

related with $\mathcal{F}(t)\mathcal{F}^{-1}(s)$, are small (or at least bounded) as $s\in\Gamma_r$.

To define the path Γ_r, associated with a pair (i',j') of indices, we observe firstly that the function

$$(t,\lambda)\to \operatorname{Re}\bigl\{\lambda\Delta_{i'j'}(t)\bigr\}$$

has constant sign for (t,λ) form the domain $\mathcal{V}$ defined in the inequalities (2.9). If $(i',j')=(i,j)$ or $(i',j')=(k,l)$ then this property is obvious. Otherwise, this follows from assumption A1 (i.e. about the lines $\mathcal{Q}_{ij}$). The constant γ which appears in definition of the domain $\mathcal{W}$ in Eq. (2.10) is such that all $\operatorname{Re}\lambda\Delta_{i'j'}(t)$, $(i',j')\neq(i,j)$, have constant sign in $\mathcal{W}$.

We consider three possibilities:

Case 1. Suppose that $\operatorname{Re}\lambda\Delta_{i'j'}(t)<0$ in $\mathcal{V}$. Then Γ_r is a path in $\mathcal{U}$ without self-intersections from $s=0$ to $s=t$.

Case 2. Suppose that $\operatorname{Re}\lambda\Delta_{i'j'}(t)>0$ in $\mathcal{V}$ and $(i',j')\neq(i,j)$ for the domain $\mathcal{U}$. Then Γ_r is a path in $\mathcal{U}$ without self-intersections from a point $s=s_r(t,\lambda,\bar\lambda)$ to $s=t$.

Case 3. Suppose that $\operatorname{Re}\lambda\Delta_{i'j'}(t)>0$ in $\mathcal{V}$ and $(i',j')=(i,j)$. Then Γ_r is a path in $\mathcal{U}$ without self-intersections from a point $s=\tilde s_r(t,\lambda,\bar\lambda)$ to $s=t$.

The begin-point functions

$$s_r(t,\lambda,\bar\lambda)\quad\text{and}\quad \tilde s_r(t,\lambda,\bar\lambda)$$

are analytic in t, are real analytic in $(\lambda,\bar\lambda)$ and are close to the point $s=1$ for $(t,\lambda)\in\mathcal{V}$. Moreover, for $s\in\Gamma_r$ the following property holds true:

$$\operatorname{Re}\lambda\bigl(\Delta_{i'j'}(t)-\Delta_{i'j'}(s)\bigr)<0 \tag{2.25}$$

(Case 1 and Cases 2 and 3 with $(t,\lambda)\in\mathcal{V}$),

$$\operatorname{Re}\bigl(T(t)-T(s)\bigr)(\mu_{i'}-\mu_{j'})<0 \tag{2.26}$$

(Case 2 with $(t,\lambda)\in\mathcal{W}\setminus\mathcal{V}$, where T is the fast time),

$$\operatorname{Re}\bigl(T(t)-T(s)\bigr)(\mu_i-\mu_j)=0 \tag{2.27}$$

(Case 3 with $(t,\lambda)\in\mathcal{W}\setminus\mathcal{V}$).

(Since for fixes t the limit $\lim_{\lambda\to\infty}s_r(t,\lambda,\bar\lambda)$ is not a point the function s_r cannot be holomorphic in λ^{-1}, the same is true about $\tilde s_r$.)

To make construction of the paths Γ_r possible we must choose suitable the constants δ_1 and δ_2 which appear in the definition of the domain $\mathcal{V}$. We omit details of the construction of these paths. We note only that for $(t,\lambda)\in\mathcal{W}\setminus\mathcal{V}$ the path Γ_r drawn in the $s^{1-\alpha}$-plane is a straight segment from $s_r^{1-\alpha}$ (or $\tilde s_r^{1-\alpha}$) to $t^{1-\alpha}$; for $(t,\lambda)\in\mathcal{V}$ the path Γ_r can be more complicated due to the fact that the lines $\mathcal{Q}_{ij}$ (see Eq. (2.4)) are generally not straight lines.

Having constructed the paths Γ_r we can come back to equation (2.24) which we treat as a fixed point equation

$$X=\mathcal{T}(X)$$

in some space $\mathcal{X}$ of functions $X(t;\lambda,\bar\lambda)$, analytic in t and real analytic in $(\lambda,\bar\lambda)$. One must equip this space with a norm which gives $\mathcal{X}$ structure of a Banach space such that the operator $\mathcal{T}$ is a contraction on some domain of $\mathcal{X}$. This norm is analogous to norms introduced in proofs of analogous theorems given in the Wasow's book [17].

In the domain $\mathcal{V}$ the situation is rather standard due to the smallness of the factor $\mathcal{F}(t)\mathcal{F}^{-1}(s)$ in Eq. (2.24) (see the property (2.25)) and also due to some bounds onto the term $F(s,X(s))$ in Eq. (2.24).

In the domain $\mathcal{W}$ we use assumption A2 which implies that there the original differential system is close to the system

$$\frac{dx}{dT}=\operatorname{diag}(\mu_1,\dots,\mu_n)\cdot x. \tag{2.28}$$

In fact, we can treat system (1.1) with the fast time T as a system with small parameter λ^{-1} and the integral equation (2.24) can be replaced with the integral equation

$$X(T)=\int_\Gamma \operatorname{diag}\bigl\{(\mu_{i'}-\mu_{j'})(T-S)\bigr\}\cdot\widetilde F\bigl(S,X(S)\bigr)\,dS. \tag{2.29}$$

Here we treat X as a function of the fast time and the collection Γ of contours is considered in the complex S-plane. The choice of the contours Γ_r is analogous to the choice from the Wasow's book, but the begin point $\widetilde{S}_r$ does no lie at infinity in Case 3. The existence of a solution to the integral equation (2.29) is proved also like in the Wasow's book and the bounds depend on the constants ε, N and Λ from definition of the domain $\mathcal{W}$ in Eq. (2.10).

Note that since the (integral) fixed point equations (2.24) and (2.29) depend on $(\lambda, \bar{\lambda})$ in real analytic way (through the real analytic dependence of the contours Γ_r), also the solution $X(t; \lambda, \bar{\lambda})$ depends on the parameter λ in real analytic way.

Let us discuss the second statement of Theorem 1, which concerns normalization in the domain $\mathcal{V}$. In this domain we can define contours Γ_r as depending analytically on t as well as on the parameter λ^{-1}. Indeed, in Case 1 this is true by definition. In the case when $\operatorname{Re} \lambda \Delta_{i'j'}(t) > 0$ in $\mathcal{V}$ we take the path Γ_r^{an} as starting from the point $s = 1$ and ending at $s = t$. With such choices the solution $X^{\mathrm{an}}(t; \lambda)$ to the integral equation (2.24) becomes complex analytic in both arguments.

The difference between the solutions $X(t; \lambda, \bar{\lambda})$ and $X^{\mathrm{an}}(t; \lambda)$ arises from the difference of the paths Γ_r and Γ_r^{an}. We can arrange these paths in such way that they differ only along some initial parts, near the point $s = 1$. Then the corresponding contributions arising from that part in the integral $\int_\Gamma \mathcal{F}(t)\mathcal{F}^{-1}(s)F(s, X(s))\,ds$ differ by a quantity of order $O(e^{-\mathrm{const}\cdot|\lambda|})$. $\square$

Remark 1. The essence of the Stokes phenomenon relies in difference of choice of the paths Γ_r in the proof of the normalization theorem; this difference occurs in Case 3 in the above proof. This can be illustrated in the following example from [23]:

$$t^{k+1}\dot{x} = \begin{pmatrix} -k & t^{k+1}\varphi(t) \\ 0 & at \end{pmatrix} x \tag{2.30}$$

(where $\varphi(t)$ is an analytic germ) with general solution $x_1(t) = De^{1/t^k} + Ce^{1/t^k} \int_{\gamma(t)} s^a e^{-1/s^k} \varphi(s)\,ds$, $x_2(t) = Ct^a$. Depending on choice of the integration path $\gamma(t)$ (which ends at $s = t$) we have different branches of the solution. Generally, one puts $D = 0$ and the path $\gamma(t)$ is such that it begins at $s = 0$, but in a sector (of the s-plane) where the term e^{-1/s^k} is small. There are k such sectors of 'fall' and k corresponding paths $\gamma_j(t)$.

The sectorial normalizations are of the form $x_1 = y_1 + \eta(t)y_2$, $x_2 = y_2$, where $\eta(t) = t^{-a}e^{1/s^k} \int_{\gamma(t)} s^a e^{-1/s^k} \varphi(s)\,ds$. In fact, we have different normalizations defined by means of different functions η_j related with different paths $\gamma_j(t)$. The Stokes matrices are of the form $\left(\begin{smallmatrix} 1 & A_j \\ 0 & 1 \end{smallmatrix}\right)$ where A_j are integrals of $s^a e^{-1/s^k} \varphi(s)$ along loops which begin and end at $s = 0$ in adjacent sectors of fall.

Example 1. Consider the third order hypergeometric equation (1.7), i.e. $(1 - t)\partial_t t \partial_t t \partial_t x + \lambda^3 x = 0$. It has two finite singular points, at $t = 0$ and at $t = 1$. In [21] the WKB solutions $g^\sigma(t; \lambda) = e^{\sigma S_3(t)}\{((1 - t)/t)^{1/3} \frac{1}{x} + \cdots\}$, $\sigma = -1$, $\epsilon = e^{i\pi/3}$, $\bar{\epsilon}$, $S_3(t) = \int_0^t \tau^{-2/3}(1 - \tau)^{-1/3}\,d\tau$, are studied and the Stokes matrices are calculated. The Stokes matrices were computed by keeping t real (in the interval $(0, 1)$) and using division of the λ-plane into sectors by means of the following 'lines of division': $\arg \lambda = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$. All these matrices are expressed via one 'principal matrix'

$$C_{21} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.31}$$

(in the basis $\{g^\sigma\}$) related with the sector between the lines $\arg \lambda = 0$ and $\arg \lambda = \frac{\pi}{3}$. The same Stokes matrices govern the Bessel like equation $\partial t \partial t \partial x + \lambda^3 x = 0$; we justify this statement in Theorem 2 below. Note also that the leading term of the corresponding linear 3-dimensional system has the form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda^3/t^2(1 - t) & 0 & 0 \end{pmatrix} x$$

and the corresponding diagonal matrix has entries $\sigma \lambda t^{-2/3}(1 - t)^{-1/3}$, $\sigma = -1, \epsilon, \bar{\epsilon}$, i.e. like in the assumptions of Theorem 1.

The hypergeometric equation (1.7) has also other system of WKB solutions: $h^\sigma(s; \lambda) = (\sigma x)^{3/2} e^{-\sigma x S_3(1)} g^\sigma(1 - s; x)$, $\sigma = -1, \epsilon, \bar{\epsilon}$, related with the singular point $t = 1 - s = 1$. Analogous calculation of the Stokes phenomenon shows that the corresponding 'principal matrix' has the form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.32}$$

We see that the Stokes phenomena related with different singular points are essentially different and it is hard to find any useful relation between them.

Remark 2. The statement of Theorem 1 can be generalized. Namely, let the matrix A be of the form

$$A(t;\lambda)=\lambda^k t^{-\alpha}\cdot \operatorname{diag}\bigl(b_1(t),\dots,b_n(t)\bigr)+O\bigl(\lambda^{k-1}\bigr)\quad \text{as } \lambda\to\infty, \tag{2.33}$$

where we assume that k is a positive integer (compare Eq. (2.1)). In this case the normal form (2.12) is generalized to a diagonal system

$$\dot y_j=\bigl(\lambda^k b_j(t)+O\bigl(\lambda^{k-1}\bigr)\bigr)y_j,\quad j=1,\dots,n.$$

The Stokes operators are defined in the same way as above.

Sometimes, e.g. in applications, one arrives to situations where the exponent k is a ratio p/q. Then one makes the standard change

$$\lambda=\mu^q \tag{2.34}$$

and obtains normalizations in sectors about $\mu=\infty$ in the μ-plane.

3. Equivalence with a Bessel like equation

In this section we deal with higher order differential equation (1.8) rather than with systems. Therefore we consider germs of differential equations

$$\partial^n x+a_1(t;\lambda)\partial^{n-1}x+\dots+a_n(t;\lambda)x=0, \tag{3.1}$$

where $\partial=\partial_t=\partial/\partial t$ and we make the following assumptions about the coefficients a_i.

They are meromorphic in t (near 0) and in λ^{-1} (near 0) and of the form

$$a_i(t;\lambda)=\frac{\nu_i}{t^i}+\sum_{j,k}a_{j,k}t^j\lambda^k, \tag{3.2}$$

where the coefficients ν_j are constant, and we have the following inequalities

$$\begin{aligned} &j>-i,\\ &k-\alpha j\leqslant \alpha i \end{aligned} \tag{3.3}$$

for indices in the sum in Eq. (3.2). Above

$$\alpha=p/q$$

is a fixed positive rational number. Moreover, we assume that:

$$\text{the number of terms with } k-\alpha j=\alpha i \text{ is finite.} \tag{3.4}$$

The first inequality in Eqs. (3.3) means that the *indicial equation*

$$\gamma(\gamma-1)\dots(\gamma-n+1)+\nu_1\gamma\dots(\gamma-n+2)+\dots+\nu_n=0 \tag{3.5}$$

for the exponents of the solutions $x\sim Ct^\gamma$ near $t=0$ does not depend on λ.

The second inequality in (3.3) means that, after introduction the *fast time*

$$T=t\lambda^\alpha, \tag{3.6}$$

we obtain an equation close to the equation

$$\partial_T^n y+b_1(T)\partial_T^{n-1}y+\dots+b_n(T)y=0 \tag{3.7}$$

(where $b_j(T)=\nu_j T^{-j}+\cdots$ are Laurent polynomials, due to the property (3.4)), with regular singularity at $T=0$, irregular singularity at $T=\infty$ and non-singular in $\mathbb{C}\setminus 0$. The other terms in Eq. (3.1) (rewritten using the fast time) will be of order $O(\lambda^{-\beta})$ for some positive (and rational) exponent β.

We call equations of the form (3.7) the *Bessel like equations*.

Theorem 2. *Under the above assumptions Eq.* (3.1) *is equivalent with Eq.* (3.7) *via a change between the corresponding fundamental matrices of solutions. Moreover, the matrix of this change is analytic in t and λ^{-1}.*

Proof. Let $\mathcal{F}(t;\lambda)$ and $\mathcal{G}(t;\lambda)$ be some fundamental matrices associated with Eqs. (3.1) and (3.7). They are analytic multivalued in t. By the assumptions the corresponding indicial equations at the regular singularity $t=0$ are the same as Eq. (3.5).

What is more important, the matrices $\mathcal{F}(t;\lambda)$ and $\mathcal{G}(t;\lambda)$ are holomorphic and single valued in λ for $\lambda \neq \infty$, but at $\lambda=\infty$ they have an essential singularity. This follows from the fact that for $C_1 < |\lambda| < C_2$ and $|t|$ small enough we can expand solutions in convergent standard Dulac type expansions (also with logarithms) with coefficients depending analytically on λ.

Let $\mathcal{F}_0(t)$ be the fundamental matrix of the corresponding *Euler equation*

$$\partial^n z + \frac{\nu_1}{t}\partial^{n-1} z + \cdots + \frac{\nu_n}{t^n} z = 0.$$

We know that the fundamental system $\phi(t) = (\phi_1, \dots, \phi_n)$ of solutions to the Euler equation consists of functions of the form $t^\gamma \ln^k t$. In particular, the *monodromy matrix* M defined by

$$\phi\big(e^{2\pi i} t\big) = \phi(t) M$$

is well defined.

By the standard theory of linear differential equations with meromorphic coefficients the fundamental systems $\varphi(t;\lambda) = (\varphi_1, \dots, \varphi_n)$ and $\psi(t;\lambda) = (\psi_1, \dots, \psi_n)$ related with Eqs (3.1) and (3.7) can be chosen as close to $\phi(t)$ (plus terms of higher order in t) and such that related to them monodromy matrices are equal to M. We have

$$\mathcal{F}\big(e^{2\pi i} t;\lambda\big) = \mathcal{F}(t;\lambda) M, \qquad \mathcal{G}\big(e^{2\pi i} t;\lambda\big) = \mathcal{G}(t;\lambda) M.$$

Therefore the matrix

$$H(t;\lambda) = \mathcal{F}(t;\lambda)\mathcal{G}^{-1}(t;\lambda)$$

is holomorphic and single valued in t (for $|t| < \varepsilon$) and in λ, e.g. for finite $|\lambda|$.

We shall prove also that:

$$\big\| H(t,\lambda) \big\| \textit{ is bounded as } |t| \textit{ is small}, |\lambda| \textit{ is large and } |T| \textit{ is large}.$$

Indeed, as $|T|$ is large the fundamental solutions to Eq. (3.7) can be chosen like the fundamental solutions of a linear equation near an irregular singular point. By the formal normal form theory they can be taken in the form

$$\tilde{\psi}_j(T) \sim e^{P_j(T^{1/b})} T^{\gamma_j}, \tag{3.8}$$

where P_j are some polynomials and b is a 'ramification index'. In the generic case we have $P_j(T^{1/b}) = \mu_j T^{k/b} + l.o.t.$ with different and nonzero coefficients μ_j, but in non-generic cases the leading monomials in P_j can coincide (see also the next section). Anyway, in sectors (of 'growth' and 'fall') about $T=\infty$ the solutions $\tilde{\psi}_j$ are well defined.

On the other hand, Eq. (3.1), rewritten using the fast time T, is a small perturbation of Eq. (3.7). So it has fundamental solutions $\tilde{\varphi}_j(T;\lambda^{-1/q})$ of the same form as in Eq. (3.8), i.e. with the same polynomials P_j and exponents γ_j. Therefore the corresponding fundamental matrices $\widetilde{\mathcal{F}}(T;\lambda^{-\gamma})$ and $\widetilde{\mathcal{G}}(T)$ are close in sectors with bounded $\|\widetilde{\mathcal{F}}\widetilde{\mathcal{G}}^{-1}\|$. Therefore also the matrix H is uniformly bounded for small $|t|$ and large $|\lambda|$. To be precise, if we assume

$$|t| < \varepsilon, \qquad |\lambda| > \Lambda = \varepsilon^{-1/\alpha},$$

with $\varepsilon > 0$ sufficiently small, then the above boundness property holds true.

It follows that the change

$$\begin{pmatrix} x \\ \dot{x} \\ \dots \\ x^{(n-1)} \end{pmatrix} = H(t) \begin{pmatrix} y \\ \dot{y} \\ \dots \\ y^{(n-1)} \end{pmatrix}$$

satisfies the thesis of Theorem 2. □

Example 2. The following equation

$$\ddot{x} = \big\{\lambda^2 t\varphi(t) + \lambda\psi\big(t,\lambda^{-1}\big)\big\}x, \tag{3.9}$$

where φ and ψ are analytic germs and $\varphi(0)=1$, is studied in [17, Eq. (29.6)]. This equation satisfies the assumptions of Theorem 2. Indeed, the corresponding Euler equation is $\ddot{z}=0$ with the fundamental matrix $\mathcal{F}_0 = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. The corresponding Bessel type equation is the *Airy equation*

$$\partial_T^2 y = Ty, \tag{3.10}$$

where the fast time equals $T = t\lambda^{2/3}$. The fundamental solutions to the Airy equation are expressed via Bessel functions:

$$\psi_1(T) = 1 + \sum_{n=1}^{\infty} \frac{(T^3/9)^n}{(2/3)_n n!} = \Gamma(2/3)\big(2T^3/9\big)^{1/3} I_{-1/3}\big(2T^{3/2}/3\big),$$

$$\psi_2(T) = T\left\{1 + \sum_{n=1}^{\infty} \frac{(T^3/9)^n}{(4/3)_n n!}\right\} = \Gamma(4/3)T\big(2T^3/9\big)^{-1/3} I_{1/3}\big(2T^{3/2}/3\big),$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol and $I_\nu(z) = \sum_{n=0}^{\infty}(z/2)^{\nu+2n}/\Gamma(\nu+n+1)n!$ are the Bessel functions (of imaginary argument).

Example 3. Consider Eq. (1.4) from the Introduction. Here the corresponding Euler equation is

$$P_0 z = 0, \qquad P_0 = \partial Q^{a_1-1}\partial Q^{a_2-1}\dots\partial Q^{a_k-1},$$

and the corresponding Bessel like equation is

$$P_0 y + y = 0 \quad \big(\text{with } T = \lambda^{a/k} t\big); \tag{3.11}$$

compare also Example 1.

When we consider Eq. (1.4) in neighborhood of $s = 1 - t = 0$, then we find the Euler equation $P_1 z = 0$, where $P_1 = s\partial_s^{a_1} s\partial_s^{a_2}\dots s\partial_s^{a_k}$, and the corresponding Bessel like equation is $P_1 y + (-1)^a y = 0$.

Theorem 2 is very essential in analysis (in [20] and [21]) of the Stokes phenomena for WKB solutions associated with the hypergeometric equations (1.6) and (1.7).

There is one important consequences of Theorem 2. Consider Eq. (3.7), about which we assume that it has irregular singularity at $T = \infty$ and that $P_j = \mu_j T^{1+\delta} + l.o.t.$ with different and nonzero leading coefficients in Eq. (3.8) (the generic case).

It is easy to find relations satisfied by the exponent δ and the coefficients μ_j. We have

$$\delta = \max\left\{\frac{1}{j}\operatorname{ord}_{T=\infty} b_j(T)\right\}. \tag{3.12}$$

Assume that

$$b_j(T) = \kappa_j T^{j\delta} + l.o.t.$$

in Eq. (3.7). Then the numbers $\tilde{\mu}_j = (1+\delta)\mu_j$ are roots of the algebraic equation

$$\tilde{\mu}^n + \kappa_1\tilde{\mu}^{n-1} + \dots + \kappa_n = 0. \tag{3.13}$$

There exist more general situations (like in the above examples), where the differential equation (3.1) is defined for t from a relatively big domain $\mathcal{D}$ in the complex plane (like in Section 2). We assume the existence of WKB solutions

$$x = \omega_j(t;\lambda) \sim e^{\lambda^d S_j(t)} \tag{3.14}$$

(where $d = \alpha(1+\delta)$), such the 'actions' $S_j(t) = \mu_j t^{1+\delta} + h.o.t.$ satisfy assumption A1 from the previous section. By results of the previous section (Theorem 1 and Remark 2) we can define Stoke operators and Stokes matrices for these WKB solutions. Recall that in general the Stokes matrices depend on $(\lambda, \bar{\lambda})$ in real analytic way. Now we can say more.

Corollary 1. *The Stokes matrices related with the WKB solutions* (3.14) *for Eq.* (3.1) *defined for $t \in \mathcal{D}$ and $|\lambda| > \Lambda$, with local data like in the assumptions of Theorem* 2*, are equivalent to constant matrices* (*as functions of* λ).

4. Formal generalized WKB expansions

In this section we consider only formal WKB expansions and we restrict our attention on solutions of nth order differential equations, like Eq. (3.1).

4.1. WKB Ansatz in generic case

In the case of Eq. (3.1) the corresponding WKB solutions are not as simple as in Introduction. Generally they are of the form

$$x = \omega(t;\lambda) \sim \exp\left\{\sum_{j=1}^{k} S_j(t)\lambda^j\right\} \times \left\{\psi_0(t) + \psi_1(t)\lambda^{-1} + \cdots\right\}, \tag{4.1}$$

where $S_j(t)$ are the *actions*, among which $S_k(t)$ is the *principal action*, and $\psi_j(t)$ are *amplitudes*.

It is easy to check that this can occur when the coefficients of Eq. (3.1) are of the form

$$a_j(t;\lambda) = a_{j,kj}(t)\lambda^{kj} + a_{j,kj-1}(t)\lambda^{kj-1} + \cdots. \tag{4.2}$$

Substituting the formal function from the right hand side of Eq. (4.1) to Eq. (3.1) we arrive to a system of equations for S_j and ψ_j. Before writing it down, we introduce some notations.

We firstly put

$$s_j(t) = \dot{S}_j(t).$$

Let

$$Z(z,\lambda) = z_1\lambda + \cdots + z_k\lambda^k, \quad z = (z_1, \ldots z_k).$$

The following identity

$$\begin{aligned} &Z^n(z,\lambda) + a_1(t;\lambda)Z^{n-1}(z,\lambda) + \cdots + a_n(t;\lambda) \\ &\quad = P_0(z_k;t)\lambda^{nk} + P_1(z_k, z_{k-1}, t)\lambda^{nk-1} + \cdots + P_{k-1}(z;t)\lambda^{nk-k+1} + \cdots \end{aligned}$$

defines uniquely polynomials P_j. They are of the form

$$\begin{aligned} &P_0 = \sum_{i=1}^{n} a_{i,ki}(t) z_k^{n-i}, \\ &P_j = \frac{\partial P_0}{\partial z_k} \cdot z_{k-j} + \widetilde{P}_j(z_k, \ldots, z_{k-j+1}; t), \quad j = 1, \ldots, k-1. \end{aligned}$$

The promised system of equations consists of the *Hamilton–Jacobi equations*

$$P_0\big(s_k(t);t\big) = 0, \tag{4.3}$$

$$P_j\big(s_k(t), \ldots, s_{k-j}(t);t\big) = 0, \quad j \geqslant 1, \tag{4.4}$$

and of the *transport equations*

$$Q(r)\dot{\psi}_0(t) + R(t)\psi_0(t) = 0, \tag{4.5}$$

$$Q(t)\dot{\psi}_j(t) + R(t)\psi_j(t) = T_j(t), \quad j \geqslant 1, \tag{4.6}$$

where

$$Q(t) = \frac{\partial P_0}{\partial z_k}\big(s_k(t);t\big), \tag{4.7}$$

$$R(t) = P_k\big(s_k(t), \ldots, s_1(t);t\big) + \frac{1}{2}\frac{\partial^2 P_0}{\partial z_k^2}\big(s_k(t);t\big)\dot{s}_k(t) \tag{4.8}$$

and the non-homogeneous terms $T_j(t)$ are expressed via the actions and their derivatives as well as via the previous amplitude functions $\psi_i(t)$ and their derivatives.

The *principal Hamilton–Jacobi equation* (4.3) has n different solutions $z_k = s_{k,1}(t), \ldots, s_{k,n}(t)$, i.e. in general situation. The corresponding actions $S_{k,j}(t) = \int^t s_{k,j}$ depend on constants of integration. Having solved Eq. (4.3) the other equations (4.4) allow to find the derivatives $z_m = s_{m,j}(t)$, $m < k$, $j = 1, \ldots, n$, of the actions $S_{m,j}(t)$ recursively. Then we solve successively the transport equations (4.5) and (4.6). Here the assumption

$$s_{k,i}(t) \not\equiv s_{k,j}(t) \quad \text{for } i \neq j, \tag{4.9}$$

called the genericity assumption, implies that $Q(t) \not\equiv 0$, what allows to solve the transport equations. The only non-uniqueness lies in the constants of integrations. So the whole procedure of derivation of the WKB solutions works well, with only minor modification.

Note also that we admit the situation when one of the actions $S_{k,j}(t)$ is identically zero, i.e. $a_{n,kn}(t) \equiv 0$ (but $a_{n-1,k(n-1)}(t) \not\equiv 0$). In that case the corresponding WKB solution is $\sim \exp\{\lambda^{k-1} S_{k-1.i}(t) + \cdots\}$.

Let us state results of the above analysis in the following:

Proposition 1. *If a solution $s_{k,i}(t)$ to the principal Hamilton–Jacobi equation is simple, $\frac{\partial P_0}{\partial z_k}(s_{k,i}(t);t) \not\equiv 0$, then Eq.* (3.1) *has a generalized WKB solution of the form* (4.1) *with $S_k = \int^t s_{k,i}(\tau)\,d\tau$.*

It follows that, if the discriminant of the polynomial P_0, with respect to the variable z_k, does not vanish identically, then Hamilton–Jacobi equations (4.3) *and* (4.4) *and transport equations* (4.5) *are solvable and Eq.* (3.1) *has a basis of generalized WKB solutions of the form* (4.1)*.*

Example 4. For the equation

$$\ddot{x} + \left(a_2\lambda^2 + a_1\lambda + a_0\right)\dot{x} + \left(b_4\lambda^4 + \cdots + b_0\right)x = 0,$$

where $a_j = a_j(t)$ and $b_j = b_j(t)$, we have

$$P_0(z_2;t) = z_2^2 + a_2 z_2 + b_4, \qquad P_1(z_1, z_2; t) = (2z_2 + a_2)z_1 + a_1 z_2 + b_3,$$
$$Q(t) = 2s_2 + a_2, \qquad R(t) = \left(z_1^2 + a_1 z_1 + b_2\right) + \dot{s}_2.$$

4.2. Non-generic case

The principal goal of this section is to extend the derivation of the formal WKB expansions to the case when the principal Hamilton–Jacobi equation (4.3) has multiple roots.

An analogous problem for system of linear differential equations in neighborhood of an irregular singular point was considered by M. Hukuhara [7], A. Levelt [13] and H. Turritin [16]. The problem there was solved completely, by applying a series of so-called shearing transformations and admitting expansions which involve rational powers of the time (see also [8] and [23]). H. Turritin [15] studied the problem of formal asymptotic expansions for systems of linear differential equations with large parameter in the spirit of the Hukuhara–Levelt–Turritin theorem. The general method used in [15] is similar to the case without parameter, but the number of steps and subcases (when applying the shearing transformations) is very big and the resulting algorithm turns out quite complicated. Moreover Turritin does not make division of the resulting system of elementary first order ODEs into Hamilton–Jacobi equations and the transport equations.

In the case of one ODE of nth order equation (3.1) the situation is much simpler. It turns out that for each WKB type solution there is only finite number of ramified parameter changes $\lambda_i \mapsto \lambda_{i+1} = \lambda_i^{1/q_i}$, $\lambda_0 = \lambda$, such that the solutions are expressed via these λ_i; moreover, $n \geqslant q_0 > q_1 > \cdots$.

Let is consider the situation when $P_0(z_k;t) = (z_k - a(t))^l \cdot \widetilde{P}_0(z_k;t)$ and $\widetilde{P}_0(a(t);t) \not\equiv 0$. So there should exist l solutions with the principal asymptotic $e^{\lambda^k \int a(t)}$.

We apply the *shearing transformation*

$$x \to \exp\left\{-\lambda^k \int^t a(\tau)\,d\tau\right\}x, \tag{4.10}$$

result of which is such that the 'symbol' of the principal Hamilton–Jacobi operator becomes

$$P_0(z_k;t) = z_k^l \cdot \widetilde{P}_0(z_k;t), \qquad \widetilde{P}_0(0;t) \not\equiv 0. \tag{4.11}$$

In terms of the coefficients in Eq. (4.2) this means that

$$a_{n-l,k(n-l)}(t) \not\equiv 0 \equiv a_{n-l+1,k(n-l+1)(t)} \equiv \cdots \equiv a_{n,kn}(t). \tag{4.12}$$

Assume that:

the term

$$a_{n-j,k(n-j)-r_j}(t)\lambda^{k(n-j)-r_j} \not\equiv 0, \quad 0 \leqslant j \leqslant l-1, \tag{4.13}$$

in each $a_j(t;\lambda)$ in Eq. (4.2) *is the first which does not vanish identically*;

it is possible that $r_j = \infty$ if $a_j \equiv 0$, but we can assume that $r_0 < \infty$, since otherwise we would deal essentially with an equation of lower degree.

We seek solutions in the form

$$x \sim \exp\left[\lambda^{k-r} S_{k-r}(t) + \cdots\right] \times \left\{\psi_0(t) + \cdots\right\}, \tag{4.14}$$

where r is a rational number (which we shall find) and below we denote $s_{k-r}(t) = \dot{S}_{k-r}$.

Let us look at the highest degree (in λ) terms (before exp) after substitution of the expression in Eq. (4.14) into particular terms in the original differential equation (3.1). From $a_{n-j}x^{(j)}$, $l < j \leqslant n$, we get

$$O\left(\lambda^{nk-jr}\right),$$

from $a_{n-l}x^{(l)}$ we get

$$\lambda^{nk-lr} \cdot a_{n-l,k(n-l)} s_{k-r}^l \psi_0 \tag{4.15}$$

(where $a_{n-l,k(n-l)}(t) \not\equiv 0$), from $a_{n-j}x^{(j)}$, $0 \leqslant j < l$, we get

$$\lambda^{nk-(r_j+jr)} \cdot a_{n-j,k(n-j)-r_j} s_{k-r}^j \psi_0. \tag{4.16}$$

We see that the contributions from the terms arising from $x^{(j)}$, $j > l$, are small in comparison with the term in Eq. (4.15). We choose $r > 0$ as the smallest possible that at least one of the terms from Eq. (4.16) has the same order as the term in Eq. (4.15), i.e. $r_j + jr = lr$. Therefore we put

$$r \stackrel{df}{=} \min_{0 \leqslant j < l} \frac{r_j}{l-j}. \tag{4.17}$$

For example, if $r_0 = 1$ then $r = 1/l$.

We have two possibilities:

$$\textit{either} \quad r < k \quad (\text{Case } 1) \quad \textit{or} \quad r \geqslant k \quad (\text{Case } 2).$$

In Case 1 we assume that r is a rational number of the form of irreducible ratio

$$r = p/q.$$

We apply the *ramified parameter change*

$$\lambda = \mu^q$$

and replace Ansatz from Eq. (4.14) with the following one:

$$\exp\left[\mu^{kq-p} S_{kq-p}(t) + \mu^{kq-p-1} S_{kq-p-1}(t) \ldots + \mu S_1(t)\right] \times \left\{\psi_0(t) + \mu^{-1}\psi_1(t) + \cdots\right\} \tag{4.18}$$

(compare Eq. (4.1)), where S_{kq-p} is the same as S_{k-r} in Eq. (4.14). Below we write $s_j(t) = \dot{S}_j$.

The above degree counting argument leads to the equation

$$\sum_{j: r_j + jr = lr} a_{n-j,k(n-j)-r_j}(t) \cdot s_{kq-p}^j(t) = 0, \tag{4.19}$$

analogous to Eq. (4.15). We can also call it the *principal Hamilton–Jacobi equation of second kind*.

Further Hamilton–Jacobi equations of second kind are obtained analogously as Eqs. (4.3) and (4.4). With

$$Z(z,\mu) = z_{kq-p}\mu^{kq-p} + \cdots + z_1\mu,$$

we have

$$\begin{aligned} &Z^n(z,\mu) + a_1\left(t;\mu^q\right)Z^{n-1}\left(z,\mu^q\right) + \cdots + a_n\left(t;\mu^q\right) \\ &\quad = P_0(z_{kq-p};t)\mu^{knq-lp} + P_1(z_{kq-p}, z_{kl-p-1}, t)\mu^{knq-lp-1} + \cdots + P_{knq-lp-1}(z;t)\mu + \cdots \end{aligned}$$

where

$$\begin{aligned} P_0 &= \sum a_{n-j,k(n-j)-r_j}(t) z_{kq-p}^j, \\ P_j &= \frac{\partial P_0}{\partial z_{kq-p}} z_{kq-p-j} + \widetilde{P}_j(z_{kq-p}, \ldots, z_{kq-p-j+1}; t) \end{aligned} \tag{4.20}$$

(and the summations runs like in Eq. (4.19)). (For the sake of simplicity we use the same notations for the polynomials P_j as in the beginning of this section.) So the *Hamilton–Jacobi equations of second kind* are the following

$$P_j\big(s_{kq-p}(t),\dots,s_{kq-p-j}(t);t\big)=0,\quad j=0,\dots,kq-p-1. \tag{4.21}$$

The *transport equations of second kind* are of the same form as Eqs. (4.5) and (4.6) but with

$$Q(t)=\frac{\partial P_0}{\partial z_{kq-p}}\big(s_{kq-p}(t);t\big)\quad\text{and}\quad R(t)=P_{kq-p}\big(s_{kq-p}(t),\dots,s_1(t);t\big)+\frac{1}{2}\frac{\partial^2 P_0}{\partial z_{kq-p}^2}\big(s_{kq-p}(t);t\big)\dot{s}_{kq-p}.$$

For any solution $z_{kq-p}=s_{kq-p,i}(t)$ to the principal Hamilton–Jacobi equation (of second kind) $P_0(z_{kq-p};t)=0$ which is simple, $\frac{\partial P_0}{\partial z_{kq-p}}(s_{kq-p,i}(t);t)\not\equiv 0$, we can solve the further Hamilton–Jacobi equations (4.21) as well as the corresponding transport equations and get a suitable WKB type solution, which is unique modulo integrations constants (see Proposition 1).

If the polynomial $P_0(z_{kq-p};t)=a_{n-l,k(n-l)}z_{kq-p}^l+l.o.t.$, which has degree l with respect to z_{kq-p}, has l distinct solutions then this allows us to find all solutions to Eq. (3.1) of the form (4.1) with fixed the first summand in the exponent in Eq. (4.1).

Example 5. Consider Eq. (1.24), i.e. $t^2(1-t)^2\ddot{x}-2\lambda t(1-t)\dot{x}+\lambda^2 x=0$ for large λ; it is a Riemann P-equation with regular singular points at $t=0,1,\infty$. Here the principal Hamilton–Jacobi equation is $[t(1-t)s_1(t)-1]^2=0$ with unique solution $s_1(t)=\frac{d}{dt}\ln\frac{t}{1-t}$. Applying the shearing change $x=(\frac{t}{1-t})^\lambda y$ we arrive to the equation

$$t^2(1-t)^2\ddot{y}+\lambda(2t-1)y=0.$$

Here the ramified change of parameter is $\mu=\lambda^{1/2}$ and the principal Hamilton–Jacobi equation (of second kind) is $[t(1-t)s_{1/2}]^2+2t-1=0$. We find $S_{1/2}=\pm\ln[(\frac{u+1}{u-1})^i\frac{u-i}{u+i}]$, where $i=e^{i\pi/2}$ and $u=\sqrt{2t-1}$, and $\psi_0=\frac{\sqrt{t(1-t)}}{2t-1}$. So Eq. (1.20) has the following generalized WKB solutions

$$x=\omega^\pm(t;\lambda)\sim\left(\frac{t}{1-t}\right)^\lambda\left(\frac{u-1}{u+1}\right)^{\pm i\sqrt{\lambda}}\left(\frac{u-i}{u+i}\right)^{\pm\sqrt{\lambda}}\times\left\{\frac{\sqrt{t(1-t)}}{2t-1}+O\big(\lambda^{-1/2}\big)\right\}.$$

If the polynomial $P_0(\cdot,t)$ in Eqs. (4.20) admits a multiple solution $z_{kq-p}=b(t)$ (i.e. for all t) then we repeat the procedure from this subsection: a shearing transformation and a ramified change of parameter. But here we have one observation.

Lemma 1. *The multiplicity l_1 of the solution $z_{kq-p}=b(t)$ is either strictly smaller than the multiplicity l of the solution we started with in this subsection or $q=1$, i.e. the number r from Eq.* (4.17) *is an integer.*

Proof. Indeed, the multiplicity of the eventual solution $z_{kq-p}=0$ is $<l$, as there are at least two monomials in the polynomial P_0. Supposition that a nonzero solution has multiplicity l leads to the non-vanishing of all the possible monomials in P_0. Then Eq. (4.17) implies the equalities $r_j=(l-j)\frac{p}{q}$ for all j, where r_j are integers. Hence $q=1$. □

If the solution $b(t)$ has multiplicity l, then we apply again the shearing transformation

$$x\to e^{-\lambda^{k-1}\int b}\cdot x,$$

without ramified change of parameter, and we continue the algorithm from this subsection.

Consider Case 2. Here we have the following:

Lemma 2. *If all the numbers r_j, $j=0,\dots,l-1$, in Eq.* (4.13) *satisfy $r_j\geqslant k(l-j)$ then there exist l independent solutions which are power series in λ^{-1}.*

Proof. Here Eq. (3.1) can be written in the form

$$\lambda^{k(n-l)}\times\big\{Q_l(t,\partial)+O\big(\lambda^{-1}\big)\big\}x=0, \tag{4.22}$$

where

$$Q_l=a_{n-l,k(n-l)}\partial^l+\cdots$$

is a linear differential operator of order l. We use the Ansatz

$$x=\psi_0(t)+\psi_1(t)\lambda^{-1}+\cdots \tag{4.23}$$

and obtain the system of lth order linear equations of the form

$$Q_l \psi_0 = 0,$$
$$Q_l \psi_j = R_j(t), \quad j > 0.$$

This system admits l independent solutions. □

Remark 3. One admits $l = 1$ in the above analysis, i.e. when the considered solution $s(t) \equiv 0$ to the principal Hamilton–Jacobi equation is simple. Since we do not make any assumption about the coefficient $a_{n-1,k(n-1)}(t)$ in the operator Q_1, we cannot expect analyticity of the solutions from the thesis of Lemma 2 in t. The corresponding solution (4.23) is divergent in general. By analogy we cannot expect convergence of the solutions from the thesis of Lemma 2 when $1 < l < n$.

But when $l = n$ Eq. (4.22) has analytic right hand side and its solutions are also analytic in λ^{-1}.

After applying several times the 'algorithm' which we described above we arrive to the following fundamental result of this section.

Theorem 3. *Any nth order equation* (3.1) *with coefficients of the form* (4.2) *with* $k \geqslant 1$ *has n independent WKB solutions of the form*

$$x \sim \exp\left\{ \sum_{j=1}^{N} S_{\alpha_j}(t)\lambda^{\alpha_j} \right\} \times \left\{ \psi_0(t) + \psi_1(t)\lambda^{-1/q} + \cdots \right\}, \tag{4.24}$$

where either $N = 0$ (*no exponential factor*) *or* α_j *are rational numbers such that*

$$\alpha_1 = \frac{p_1}{q_1} > \alpha_2 = \frac{p_2}{q_1 q_2} > \cdots > \alpha_N = \frac{p_N}{q_1 \ldots q_N} = \frac{p_N}{q} > 0,$$

where p_j *and* q_j *are positive integers,* $q_1 \leqslant n$ *and* $q_j < \max\{q_i \colon\ i < j,\ q_i > 1\}$. *The actions* S_{α_j} *and the amplitudes* ψ_j *in Eq.* (4.24) *are defined algorithmically and uniquely modulo constants of integration.*

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